New special functions solving nonlinear autonomous dynamical systems

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Abstract

A general solution is found for a large class of time continuous autonomous nonlinear dynamical systems, the so-called quasi-polynomial systems. This solution is expressed in terms of a new type of special functions defined via their Taylor series. The coefficients of these Taylor series are generated by a tensor that generalizes the factorial function and has a combinatorial meaning. The existence of these functions raises the question of the relation between them and the chaotic behaviour of the solutions that may appear for the quasi-polynomial dynamical systems.

Introduction

This article is intended to draw the attention of specialists in the fields of special functions, combinatorics and nonlinear dynamics on a new class of special functions that solve a broad class of nonlinear dynamical systems.

Let us define an autonomous dynamical system in continuous time as a system of coupled nonlinear ODEs of the form

$$\dot{x}_i = f_i(x_1, \cdots, x_n) \tag{1}$$

where the dot denotes the time derivative, the index i runs from 1 to n, the dependent variables x_i are real, the functions f_i are enough regular to ensure

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the existence and unicity of solutions, and the boundary conditions are given at the initial time.

For such systems there does not exist a universal structure of the general solution. This is in strong contrast with the particular case of linear autonomous dynamical systems for which the general solution of the Cauchy problem is given by the exponential of the constant matrix that characterizes each of these systems.

One reason for this lack of universality is the infinite diversity of possible functional forms of the functions f_i in equations (1). Moreover, the nonlinearity of the equations entails a factorial explosion in the coefficients of the Taylor series of the solutions: A coefficient of order k involves a sum of $k!$ terms that depend on the specific form of the functions f_i. The recursion relations between these coefficients are as difficult to solve as the original ODEs. Hence, one should not wonder that in most cases it is impossible to find a closed form structure for the general k -th order Taylor coefficient of such solutions.

In this article, we show that for the class of quasi-polynomial systems defined in the next chapter, a universal form of the general solution to the Cauchy problem can be found. Moreover, as has been shown by E.Kerner [7], most systems of non-polynomial nonlinear ODEs relevant for physics, chemical kinetics and generally for mathematical modeling in natural and social sciences can be reshaped in the form of systems of polynomial ODEs. The latter, in turn, are a sub-class of the quasi-polynomial systems of ODEs. This general property, thus, extends our general solution to most nonlinear systems of ODEs defining dynamical systems.

The quasi-polynomial dynamical systems

Let us define the sub-class of dynamical systems (1) for which the functions f_i are quasi-polynomials, i.e. finite sums of monomials involving powers of the dependent variables that can be real numbers. Such monomials we shall call quasi-monomials.

The systems of this class can be cast into a useful standard notation, the

so-called quasi-polynomial (QP) representation[1, 2]:

$$\dot{x}_i = x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad \text{for } i = 1, \ldots, n \tag{2}$$

where N refers to the number of quasi-monomials in the set of variables x_j in the right-hand side, and A and B are constant rectangular matrices respectively $n \times N$ and $N \times n$ with entries that are real numbers. The presence of the factor x_i in front of the right-hand side of the above equation is essential and constrains the definition of both matrices. However, it does not restrict the generality of the class of systems (2) as any polynomial or quasi-polynomial can be represented in that form. The presence of the factor x_i underlines the importance of the logarithmic time derivative of the functions $x_i(t)$ which plays a fundamental role in the present approach.

Other authors have found independently the same form (2) and derived from it theoretical results in the fields of chemical reactions and ecological systems[3, 4, 5]. These works were mainly concerned with models describing complex networks of interacting entities and they were essentially devoted to the study of the stability properties of some of their solutions. More generally, the QP class covers almost all the systems of interest for mathematical modeling in natural and social sciences.

A fundamental feature of the above QP representation of a dynamical system is its covariance under the group of quasi-monomial transformations. These are transformations of the dependent variables defined as follows:

$$x_i = \prod_{k=1}^{n} \tilde{x}_k^{C_{ik}} \qquad \text{for } i = 1, \ldots, n \tag{3}$$

where the matrix C is any invertible, constant, real, square $n \times n$ matrix. In order to avoid unrelevant discussions about the existence of the inverses of these transformations and their differentiability we limit our scope to systems and to initial conditions such that their solutions remain in the positive cone. However, it can be proven that most of the following results continue to be valid for systems not fulfilling this restriction.

It can be easily shown that under transformations (3) the system (2)

becomes:

$$\dot{\tilde{x}}_i = \tilde{x}_i \sum_{j=1}^{N} \tilde{A}_{ij} \prod_{k=1}^{n} \tilde{x}_k^{\tilde{B}_{jk}} \qquad \text{for } i = 1, \ldots, n \tag{4}$$

with the following rule of transformations for the matrices A and B:

$$\tilde{A} = C^{-1} A \tag{5}$$

and

$$\tilde{B} = BC \tag{6}$$

where the products are matrix products. The identity of form of equations (2) and (4) clearly exhibits the covariance of the QP equations under the quasi-monomial group of transformations. Let us stress that:

$$\tilde{B}\tilde{A} = BA \tag{7}$$

in other words, the $N \times N$ matrix BA is an invariant of the quasi-monomial transformations (3). This means that the whole set of QP-systems is divided into equivalence classes labeled by such $N \times N$ matrices. The fundamental matrix BA is related to the existence of a canonical form in each equivalence class as we now show.

Indeed, under a particular quasi-monomial transformation, any QP system can be brought to a canonical form, the so-called Lotka-Volterra system of ODEs [1, 2, 6]:

$$\dot{x}_i = x_i \sum_{j=1}^{N} M_{ij} x_j \qquad \text{for } i = 1, \ldots, N \tag{8}$$

where N is the number of monomials of the original QP system (2) and the square $N \times N$ matrix M is equal to the matricial product BA of the two matrices B and A. There also exists a second canonical form dual of the above one, which until now has been much less studied [1] but is also characterized by the same matrix BA.

In a given equivalence class, the passage to the Lotka-Volterra form is most easily shown in the particular case of QP systems for which the number N of quasi-monomials is equal to the dimension n of the system (2), the so-called square QP-systems. For the square QP-systems the matrices A and

B are of course square matrices. In this case, and if the matrix B is not singular, the quasi-monomial transformation (3) in which the matrix C is chosen equal to B^{-1} leads to the equation (4) with

$$\tilde{A} = BA \tag{9}$$

and

$$\tilde{B} = I \tag{10}$$

where I is the identity matrix. Thus, the transformed equation is now:

$$\dot{x}_i = x_i \sum_{j=1}^{N} (BA)_{ij} x_j \qquad \text{for } i = 1, \ldots, N \tag{11}$$

where we omitted for convenience the tilde accent on the new variables x_i in the transformed equation. This is exactly the Lotka-Volterra form announced in (8) with $M = BA$, i.e. M is the invariant matrix (7).

For the non-square QP systems, i.e. the systems whose matrices A and B respectively are $n \times N$ and $N \times n$, the Lotka-Volterra form (8) is also shown to be the canonical form with a matrix $M = BA$ that is $N \times N$ [2, 6]. Thus, the reduction to the Lotka-Volterra canonical form is a general property of the QP dynamical systems.

The fact that the transformation leading to the canonical form is a diffeomorphism reduces the analysis of the solutions of general QP-systems to those of the corresponding Lotka-Volterra systems. This result paves the way to the transfer of the whole corpus of knowledge related to the latter equations which is pretty extensive.

We now show that this reduction to the Lotka-Volterra format leads to an explicit formula for the general solution of the QP-systems. This solution is expressed in terms of a Taylor series whose general coefficient is explicitely known.

General solution of QP-systems

The Taylor series in time t for the solution $x_i(t)$ of the Lotka-Volterra differential system (8) is defined as

$$x_i(t) = \sum_{k=0}^{\infty} c_i(k) \frac{t^k}{k!} \qquad \text{for } i = 1, \ldots, N \tag{12}$$

The coefficient $c_i(k)$ is calculated by performing the k order time derivative of $x_i(t)$ at time $t = 0$. This is readily done by iterating the t derivative while recursively using the Lotka-Volterra system (8). The result is amazingly simple:

$$c_i(k) = x_i(0) \sum_{i_1=0}^{N} \ldots \sum_{i_k=0}^{N} M_{ii_1} \left(M_{ii_2} + M_{i_1 i_2}\right) \ldots \left(M_{ii_k} + M_{i_1 i_k} + \cdots + M_{i_{k-1} i_k}\right) x_{i_1}(0) \ldots x_{i_k}(0) \tag{13}$$

where the $x_i(0)$ are the components of the initial condition. Here again we omitted the tilde accent on the variables x_i .

The considered Lotka-Volterra system results from the transformation (8,9,10) of a QP-system (2). Going back to the original variables of this QP-system, we get for the order k coefficient $C_i(k)$ of the Taylor series solving this system:

$$C_i(k) = x_i(0) \sum_{i_1=0}^{N} \ldots \sum_{i_k=0}^{N} A_{ii_1} \left(A_{ii_2} + M_{i_1 i_2}\right) \ldots \left(A_{ii_k} + M_{i_1 i_k} + \ldots + M_{i_{k-1} i_k}\right) \prod_{j_1=1}^{n} x_{j_1}(0)^{B_{i_1 j_1}} \ldots \prod_{j_k=1}^{n} x_{j_k}(0)^{B_{i_k j_k}} \tag{14}$$

where i runs from 1 to n and n is the dimension of the original QP-system. Thus, the general solution of the QP-system is

$$x_i(t) = \sum_{k=0}^{\infty} C_i(k) \frac{t^k}{k!} \qquad \text{for } i = 1, \ldots, n \tag{15}$$

An observation of the coefficients $c_i(k)$ clearly reveals a combinatorial structure related to the product

$$M_{ii_1} \left(M_{ii_2} + M_{i_1 i_2}\right) \ldots \left(M_{ii_k} + M_{i_1 i_k} + \ldots + M_{i_{k-1} i_k}\right) \tag{16}$$

.

For a one dimensional Lotka-Volterra system this product would reduce to $k! M^k$. In more dimensions the number M is replaced by the components

of the matrix M_{ij}. By doubling the number of sums and introducing the corresponding Kronecker deltas the previous product becomes

$$\sum_{j_1=0}^{N} \dots \sum_{j_k=0}^{N} \delta_{ij_1} \left(\delta_{ij_2} + \delta_{i_1 j_2}\right) \dots \left(\delta_{ij_k} + \delta_{i_1 j_k} + \dots + \delta_{i_{k-1} j_k}\right) M_{j_1 i_1} M_{j_2 i_2} \dots M_{j_k i_k} \tag{17}$$

where the tensor

$$\delta_{ij_1}(\delta_{ij_2} + \delta_{i_1 j_2}) \dots \left(\delta_{ij_k} + \delta_{i_1 j_k} + \dots + \delta_{i_{k-1} j_k}\right) \tag{18}$$

is a N-dimensional generalization of the factorial function $k!$. Its role in the structure of the Taylor coefficient is double. It governs the type of contracted products between the k tensors M that appear in the coefficient $c_i(k)$, and also counts these types. The combinatorial study of this object would certainly lead to interesting perspectives.

The above Taylor series (12), when it converges, can be considered as defining a new class of special functions which presents some analogy with the large class of the hypergeometric functions. In analogy with the latter class of special functions, the study of other properties of these new functions such as asymptotic series expansions and integral representations would be of the greatest interest. Indeed, the QP-systems are known to have for certain values of their parameters chaotic solutions. The latter have extremely complex behaviours in t such as the sensitivity with respect to the variations of the initial conditions or strange attractors having fractal geometry in the phase-space. These properties should be related in some way to the properties of the functions that we generated as solutions of these systems. Such a study would certainly contribute to develop a bridge between the theory of nonlinear dynamical systems and the theory of special functions.
Furthermore, we believe that the study of the combinatorial properties of the tensor (18) could be of great interest for both combinatorics and dynamical system theory.

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